Trocq: Proof Transfer for Free, With or Without Univalence

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Context

I am working with a proof assistant with a trusted kernel in DTT. (E.g. Coq, Lean, Agda, ...)

- I want to get a concrete value **within the proof assistants**, e.g. withing Coq/MATHCOMP [11]:
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Context and motivation

Some previous work answering exactly the questions above

- CoqEAL [5] (Cano, me, Dénès, Martin-Dorel, Mörtberg, Rouhling, Roux, Siles), does **data** transfer.
- Univalent Parametricity [13, 14] (Sozeau, Tabareau, Tanter), changes representation **using univalence**.

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This work generalizes both. Indeed we may

- change representation **without univalence** in some cases,
- change representation **with partial isos** in some cases.

Comparison to other previous work in the paper.

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Troc [tʁɔk] subst. masc.
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∼ "A direct exchange of goods without the use of money" CNRTL

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[Cyril Cohen, Enzo Crance, and Assia Mahboubi. "Artifact Report: Trocq: Proof Transfer for Free, With or Without Univalence". In: *ESOP 2024 - 33rd European Symposium on Programming*. Vol. LNCS-14576. Programming Languages and Systems 33rd European Symposium on Programming, ESOP 2024, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2024, Luxembourg City, Luxembourg, April 6–11, 2024, Proceedings, Part I. Luxembourg, Luxembourg: Springer Nature Switzerland, Apr. 2024, pp. 269–274. DOI: 10.1007/978-3-031-57262-3_11. URL: https://inria.hal.science/hal-04623207]

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Old and new examples

The canonical example

We have the standard definition of Peano natural numbers (stdlib)

$$\texttt{Inductive}\ \mathbb{N} := \texttt{O}_{\mathbb{N}} : \mathbb{N} \mid \texttt{S}_{\mathbb{N}}\ (\texttt{n} : \mathbb{N}) : \mathbb{N}.$$

For which, we have:

$$\mathbb{N}_{\text{ind}} : \forall\ \texttt{P} : \mathbb{N} \longrightarrow \square,\ \texttt{P}\ \texttt{O}_{\mathbb{N}} \longrightarrow (\forall\ \texttt{n} : \mathbb{N},\ \texttt{P}\ \texttt{n} \longrightarrow \texttt{P}\ (\texttt{S}\ \texttt{n})) \longrightarrow \forall\ \texttt{n} : \mathbb{N},\ \texttt{P}\ \texttt{n}$$

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ℕind : ∀ P : ℕ → □, P 0ℕ → (∀ n : ℕ, P n → P (S n)) → ∀ n : ℕ, P n
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Here is an alternative binary representation (stdlib)

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Inductive pos := xI : pos → pos | xO : pos → pos | xH : pos.
Inductive N := 0N : N | Npos : pos → N.

Fixpoint Spos (p : pos) : pos := match p with
  xH ⇒ xO xH | xO p ⇒ xI p | xI p ⇒ xO (Spos p) end.

Definition SN (n : N) :=
  match n with Npos p ⇒ Npos (Spos p) | _ ⇒ Npos xH end.
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1. Computing the degree of a polynomial, e.g. $\deg((2X + X^5 + 1)X^2) = 7$.

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Planned integration of Trocqto Coq

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decide  (* := by rewrite computable; vm_compute. *)
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(decide is the name of the equivalent lean tactic)

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by rewrite (@mod 9); decide. (* where mod : ∀{n}, Z -> Z/nZ *)
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   - or ... use directly $\mathbb{N}$ lemmas on $\mathbb{N} \cup \{-\infty\}$

2

Revisiting parametricity and univalent parametricity

Parametricity: standard version [2]

- Context translation:

$$\llbracket \langle\rangle \rrbracket = \langle\rangle \tag{1}$$

$$\llbracket \Gamma, x : A \rrbracket = \llbracket \Gamma \rrbracket, x : A, x' : A', x_R : \llbracket A \rrbracket\ x\ x' \tag{2}$$

- Term translation:

$$\llbracket \Box_i \rrbracket = \lambda A\, A'.\, A \to A' \to \Box_i \tag{3}$$

$$\llbracket x \rrbracket = x_R \tag{4}$$

$$\llbracket A\ B \rrbracket = \llbracket A \rrbracket\ B\ B'\ \llbracket B \rrbracket \tag{5}$$

$$\llbracket \lambda x : A.\, t \rrbracket = \lambda(x : A)(x' : A')(x_R : \llbracket A \rrbracket\ x\ x').\, \llbracket t \rrbracket \tag{6}$$

$$\llbracket \Pi x : A.\, B \rrbracket = \lambda f\ f'.\, \Pi(x : A)(x' : A')(x_R : \llbracket A \rrbracket\ x\ x').\ \llbracket B \rrbracket (f\ x)(f'\ x') \tag{7}$$

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$$[\![\square_i]\!] = \lambda A\, A'.\, A \to A' \to \square_i \tag{3}$$

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- Abstraction theorem: If $\Gamma \vdash t : T$ then $[\![\Gamma]\!] \vdash t : T$, $[\![\Gamma]\!] \vdash t' : T'$ and $[\![\Gamma]\!] \vdash [\![t]\!] : [\![T]\!]\ t\ t'$.

Parametricity: sequent style

Parametricity contexts:

$$\Xi ::= \varepsilon \mid \Xi,\; x \sim x' \;\because\; x_R.$$

Parametricity rules:

$$\frac{}{\Xi \vdash \square_i \sim \square_i \;\because\; \lambda(A\,B : \square_i).\, A \to B \to \square_i}\ (\textsc{ParamSort})$$

$$\frac{(x, x', x_R) \in \Xi \qquad \Xi \vdash}{\Xi \vdash x \sim x' \;\because\; x_R}\ (\textsc{ParamVar})$$

$$\frac{\Xi \vdash M \sim M' \;\because\; M_R \qquad \Xi \vdash N \sim N' \;\because\; N_R}{\Xi \vdash M\,N \sim M'\,N' \;\because\; M_R\,N\,N'\,N_R}\ (\textsc{ParamApp})$$

$$\frac{\Xi, x \sim x' \;\because\; x_R \vdash M \sim M' \;\because\; M_R}{\Xi \vdash \lambda x : A.\, M \sim \lambda x' : A'.\, M' \;\because\; \lambda x\,x'\,x_R.\, M_R}\ (\textsc{ParamLam})$$

$$\frac{\Xi \vdash A \sim A' \;\because\; A_R \qquad \Xi, x \sim x' \;\because\; x_R \vdash B \sim B' \;\because\; B_R \qquad x, x' \notin \mathrm{Var}(\Xi)}{\Xi \vdash \Pi x : A.\, B \sim \Pi x' : A'.\, B' \;\because\; \lambda f\,g.\, \Pi x\,x'\,x_R.\, B_R\ (f\ x)\ (g\ x')}\ (\textsc{ParamPi})$$

Parametricity: sequent abstraction theorem

We say Ξ is admissible for Γ if

$$\Gamma \rhd \Xi \triangleq \frac{\Xi \vdash \Gamma(x) \sim A' \because A_R}{\Gamma(x') = A' \wedge \Gamma(x_R) = A_R\ x\ x'}$$

We rephrase the abstraction theorem:

$$\frac{\Gamma \vdash \qquad \Gamma \vdash M : A \qquad \Gamma \rhd \Xi \qquad \Xi \vdash M \sim M' \because M_R \qquad \Xi \vdash A \sim A' \because A_R}{\Gamma \vdash M' : A' \qquad \text{and} \qquad \Gamma \vdash M_R : A_R\ M\ M'}$$

In particular, by applying it to $\Gamma \vdash A : \square_i$ instead, we get:

$$\frac{\Gamma \rhd \Xi \qquad \Xi \vdash A \sim A' \because A_R}{\Gamma \vdash A_R : A \to A' \to \square_i}$$

Example: motivating raw paramericity

Assume we have a derivation

$$\dfrac{\vdots}{\mathsf{fold}\ \mathbb{N}\ 0_{\mathbb{N}}\ (+_{\mathbb{N}})[1_{\mathbb{N}}, 2_{\mathbb{N}}, 3_{\mathbb{N}}] \sim \mathsf{fold}\ \mathrm{N}\ 0_{\mathrm{N}}\ (+_{\mathrm{N}})[1_{\mathrm{N}}, 2_{\mathrm{N}}, 3_{\mathrm{N}}]\ \because\ w}$$

Example: motivating raw paramericity

Assume $\phi : \mathbb{N} \to \mathrm{N}$ and

$$\begin{aligned} \mathbb{N} \sim \mathrm{N} &\because \lambda(m : \mathbb{N})(n : \mathrm{N}).\phi(m) = n \\ 0_{\mathbb{N}} \sim 0_{\mathrm{N}} &\because (0_R : \phi(0_{\mathbb{N}}) = 0_{\mathrm{N}}) \\ S_{\mathbb{N}} \sim S_{\mathrm{N}} &\because (S_R : \forall m \forall n, \phi(m) = n \to \phi(S_{\mathbb{N}} m) = S_{\mathrm{N}} n) \end{aligned}$$

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0_{\mathbb{N}} &\sim 0_{\mathrm{N}} \because (0_R : \phi(0_{\mathbb{N}}) = 0_{\mathrm{N}}) \\
S_{\mathbb{N}} &\sim S_{\mathrm{N}} \because (S_R : \forall m \forall n, \phi(m) = n \to \phi(S_{\mathbb{N}} m) = S_{\mathrm{N}} n)
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Then w has type

$$\phi\,(\mathrm{fold}\ \mathbb{N}\ 0_{\mathbb{N}}\ (+_{\mathbb{N}})[1_{\mathbb{N}}, 2_{\mathbb{N}}, 3_{\mathbb{N}}]) = \mathrm{fold}\ \mathrm{N}\ 0_{\mathrm{N}}\ (+_{\mathrm{N}})[1_{\mathrm{N}}, 2_{\mathrm{N}}, 3_{\mathrm{N}}]$$

Example: motivating univalent paramericity

Assume $P : \square_i \longrightarrow \square_j$ is a closed term.

$$P\mathbb{N} \sim P\mathbf{N} \because w$$

The witness w has type

$$[\![\square_j]\!] \, (P\mathbb{N}) \, (P\mathbf{N})$$

i.e.

$$(P\mathbb{N}) \longrightarrow (P\mathbf{N}) \longrightarrow \square_j$$

Example: motivating univalent paramericity

Assume $P : \square_i \longrightarrow \square_j$ is a closed term.

$$P\mathbb{N} \sim P\mathbf{N} \because w$$

The witness w has type

$$[\![\square_j]\!] \; (P\mathbb{N}) \; (P\mathbf{N})$$

i.e.

$$(P\mathbb{N}) \longrightarrow (P\mathbf{N}) \longrightarrow \square_j$$

We want

$$(P\mathbb{N}) \leftrightarrow (P\mathbf{N})$$

Univalent parametricity: standard version

- Term translation:

$$[\,\square_i\,] = p_{\square_i}$$

$$[\,x\,] = x_R$$

$$[\,A\ B\,] = [\,A\,]\ B\ B'\ [\,B\,]$$

$$[\,\lambda x : A.\,t\,] = \Pi(x : A)(x' : A')(x_R : [\![\,A\,]\!]\ x\ x').\,[\,t\,]$$

$$[\,\Pi x : A.\,B\,] = p_{\Pi}\ [\,A\,]\ [\,B\,]$$

- Type translation: $[\![\,A\,]\!] = [\,A\,].1 \qquad [\![\,A\,]\!]^{\text{eq}} = [\,A\,].2 \qquad [\![\,A\,]\!]^{\text{coh}} = [\,A\,].3$

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$$[\,\square_i\,] = \begin{pmatrix} \lambda A\,B.\,\Sigma(R : \mathsf{rel}_i\ A\ B)(e : A \simeq B).\Pi ab.(R\ a\ b) \simeq (a = e^{-1} b) \\ \mathsf{id}_{\square_i} \\ \mathsf{univ}_{\square_i} \end{pmatrix}$$

$$[\,x\,] = x_R$$

$$[\,A\ B\,] = [\,A\,]\ B\ B'\ [\,B\,]$$

$$[\,\lambda x : A.\,t\,] = \Pi(x : A)(x' : A')(x_R : [\![\,A\,]\!]\ x\ x').\,[\,t\,]$$

$$[\,\Pi x : A.\,B\,] = \begin{pmatrix} \lambda f\ f'.\,\Pi(x : A)(x' : A')(x_R : [\![\,A\,]\!]\ x\ x').\,[\![\,B\,]\!](f\ x)(f'\ x') \\ \mathsf{Equiv}_{\Pi}[\![\,A\,]\!]^{\mathsf{eq}}[\![\,B\,]\!]^{\mathsf{eq}} \\ \mathsf{univ}_{\Pi} \end{pmatrix}$$

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- Abstraction theorem: If $\Gamma \vdash t : T$ then $[\![\,\Gamma\,]\!] \vdash [\,t\,] : [\![\,T\,]\!]\ t\ t'$.

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- Remark A: If $\Gamma \vdash A : \square_i$ then $[\![\,\Gamma\,]\!] \vdash [\,A\,] : [\![\,\square_i\,]\!]\ A\ A'$.

Univalent parametricity: standard version

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- Remark B: $\vdash [\,\square_i\,] : [\![\,\square_{i+1}\,]\!]\ \square_i\ \square_i$.

Univalent parametricity: standard version

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$$[\,\square_i\,] = p_{\square_i}$$

$$\begin{aligned}[\,x\,] &= x_R \\ [\,A\ B\,] &= [\,A\,]\ B\ B'\ [\,B\,] \\ [\,\lambda x : A.\,t\,] &= \Pi(x : A)(x' : A')(x_R : [\![\,A\,]\!]\ x\ x').\,[\,t\,]\end{aligned}$$

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- Type translation: $[\![\,A\,]\!] = [\,A\,].1 \qquad [\![\,A\,]\!]^{\mathrm{eq}} = [\,A\,].2 \qquad [\![\,A\,]\!]^{\mathrm{coh}} = [\,A\,].3$
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Univalent parametricity: sequent style

$$\frac{}{\Xi \vdash_u \square_i \sim \square_i \because p_{\square_i}} \text{(UParamSort)}$$

$$\frac{(x, x', x_R) \in \Xi \qquad \Xi \vdash}{\Xi \vdash_u x \sim x' \because x_R} \text{(UParamVar)}$$

$$\frac{\Xi \vdash_u M \sim M' \because M_R \qquad \Xi \vdash_u N \sim N' \because N_R}{\Xi \vdash_u M\ N \sim M'\ N' \because M_R\ N\ N'\ N_R} \text{(UParamApp)}$$

$$\frac{\Xi \vdash_u A \sim A' \because A_R \qquad \Xi, x \sim x' \because x_R \vdash_u M \sim M' \because M_R}{\Xi \vdash_u \lambda x : A.\, M \sim \lambda x' : A'.\, M' \because \lambda x\, x'\, x_R.\, M_R} \text{(UParamLam)}$$

$$\frac{\Xi \vdash_u A \sim A' \because A_R \qquad \Xi, x \sim x' \because x_R \vdash_u B \sim B' \because B_R}{\Xi \vdash_u \Pi x : A.\, B \sim \Pi x' : A'.\, B' \because p_\Pi\ A_R\ B_R} \text{(UParamPi)}$$

Univalent parametricity: sequent abstraction

We rephrase the Univalent parametricity abstraction theorem:

$$\frac{\mathit{\Gamma} \vdash \qquad \mathit{\Gamma} \vdash M : A \qquad \mathit{\Gamma} \rhd \mathit{\Xi} \qquad \mathit{\Xi} \vdash_u M \sim M' \because M_R \qquad \mathit{\Xi} \vdash_u A \sim A' \because A_R}{\mathit{\Gamma} \vdash M' : A' \qquad \text{and} \qquad \mathit{\Xi} \vdash_u M_R : \left(A_R\ M\ M'\right).1}$$

Remark A:

$$\frac{\mathit{\Gamma} \vdash A : \square_i \qquad \mathit{\Xi} \vdash_u A \sim A' \because A_R \qquad \mathit{\Gamma} \rhd \mathit{\Xi}}{\mathit{\Gamma} \vdash_u A_R : \left(p_{\square_i}\ A\ A'\right).1}$$

Remark B:

$$\vdash_u p_{\square_i} : \left(p_{\square_{i+1}}\ \square_i\ \square_i\right).1$$

3

Type equivalence in a kit

Observation

The key datastructure in univalent parametricity is the one of **relations which are the graph of equivalences**

$$\boxdot^{u}\ A\ B \triangleq \left(\Sigma(R : A \to B \to \square)(e : A \simeq B).\Pi ab.(R\ a\ b) \simeq (a = e^{-1} b)\right).$$

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The two following observations

- "inhabiting this structure triggers uses of univalence",
- "it is not symmetric (one direction is privileged in e)",

Observation

The key datastructure in univalent parametricity is the one of **relations which are the graph of equivalences**

$$\boxdot^{u}\ A\ B \triangleq \left(\Sigma(R : A \to B \to \Box)(e : A \simeq B).\Pi ab.(R\ a\ b) \simeq (a = e^{-1} b)\right).$$

The two following observations

- "inhabiting this structure triggers uses of univalence",
- "it is not symmetric (one direction is privileged in e)",

correspond exactly to the two achievements:

- change representation **without univalence** in some cases,
- change representation **with partial isos** in some cases.

Disassembling type equivalence

- We use a variation of (exercise in the HoTT Book):

$$(A \simeq B) \;\simeq\; \Sigma R : A \to B \to \square.\, \mathsf{IsFun}(R) \times \mathsf{IsFun}(R^{-1})$$

$$\text{with} \quad \mathsf{IsFun}(R) \triangleq \Pi a : A.\, \mathsf{IsContr}(\Sigma b : B.\, R\ a\ b)$$

$$R^{-1} \triangleq \lambda a\ b.\, R\ b\ a$$

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- Then we remark $\mathsf{IsFun}(R) \simeq \mathsf{IsUmap}(R)$, where

$$\begin{aligned}\mathsf{IsUmap}(R) \triangleq\; & \Sigma(m : A \to B). \\ & \Sigma(g_1 : \Pi(a : A)(b : B).\, m\; a = b \to R\; a\; b). \\ & \Sigma(g_2 : \Pi(a : A)(b : B).\, R\; a\; b \to m\; a = b). \\ & \Pi(a : A)(b : B).\, (g_1\; a\; b) \circ (g_2\; a\; b) \doteqdot id.\end{aligned}$$

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- We pose

$$\boxdot^{\top}\ A\ B \;\triangleq\; \Sigma R : A \to B \to \square.\, \mathsf{IsUmap}(R) \times \mathsf{IsUmap}(R^{-1})$$

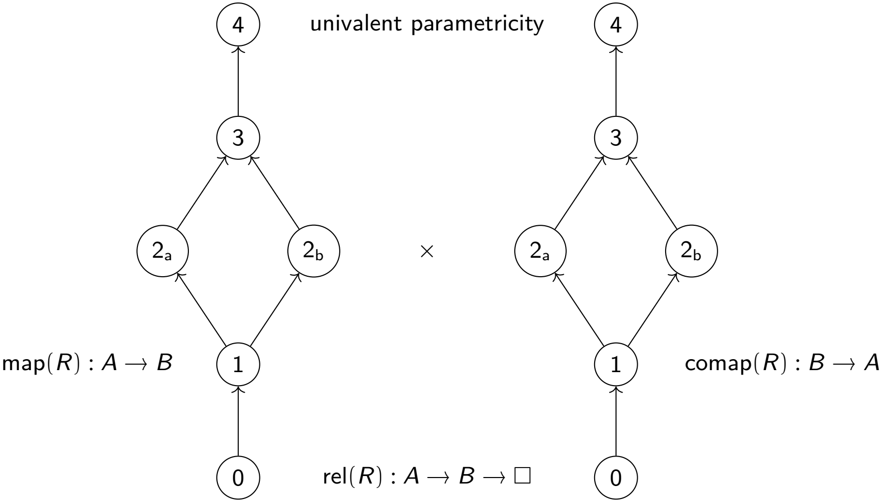
Reassembling type equivalence

For $\alpha = (n, k) \in \mathcal{A} \triangleq \{0, 1, 2_a, 2_b, 3, 4\}^2$, we pose:

$$\boxdot^{\alpha} \triangleq \lambda(A\ B\ :\ \square).\Sigma(R\ :\ A \to B \to \square).\mathrm{Class}_{\alpha}\ R$$

$$\begin{aligned}
\mathrm{Class}_{\alpha}\ R &\triangleq (\mathrm{M}_n\ R) \times (\mathrm{M}_k\ R^{-1}) \\
\mathrm{M}_0\ R &\triangleq . \\
\mathrm{M}_1\ R &\triangleq (A \to B) \\
\mathrm{M}_{2_a}\ R &\triangleq \Sigma m : A \to B.\, G_{2_a}\ m\ R \\
G_{2_a}\ m\ R &\triangleq \Pi a\, b.\, m\ a = b \to R\ a\ b \\
\mathrm{M}_{2_b}\ R &\triangleq \Sigma m : A \to B.\, G_{2_b}\ m\ R \\
G_{2_b}\ m\ R &\triangleq \Pi a\, b.\, R\ a\ b \to m\ a = b \\
\mathrm{M}_3\ R &\triangleq \Sigma m : A \to B.\, (G_{2_a}\ m\ R) \times (G_{2_b}\ m\ R) \\
\mathrm{M}_4\ R &\triangleq \Sigma m : A \to B.\, \Sigma(g_1 : G_{2_a}\ m\ R).\, \Sigma(g_2 : G_{2_b}\ m\ R). \\
&\qquad \Pi a\, b.\, (g_1\ a\ b) \circ (g_1\ a\ b) \doteqdot id
\end{aligned}$$

The lattice of annotations $\mathcal{A}$



Fun facts about $\boxdot^{\cdot}$

- Noting $\bot = (0,0)$, $\boxdot^{\bot}$ is equivalent to the data of a relation.
- Noting $\top = (4,4)$, the definitions of $\boxdot^{\top}$ and $\boxdot^{(4,4)}$ coincide.
- $\boxdot^{(4,0)}\ A\ B$ is the same as a function $A \longrightarrow B$
- $\boxdot^{(0,4)}\ A\ B$ is the same as a function $B \longrightarrow A$
- $\boxdot^{(4,2_a)}\ A\ B$ is the same as a split epi $A \twoheadrightarrow B$
- $\boxdot^{(4,2_b)}\ A\ B$ is the same as a split mono $A \rightarrowtail B$

The elements $p_{\square}^{\alpha,\beta}$ of $\boxdot^{\beta} \square \square$

Let

$$\mathcal{D}_{\square} = \{(\alpha, \beta) \in \mathcal{A}^2 \mid \alpha = \top \vee \beta \in \{0, 1, 2_a\}^2\}$$

For all $(\alpha, \beta) \in \mathcal{D}_{\square}$ we can define $p_{\square}^{\alpha,\beta}$ such that

$$\vdash_u p_{\square}^{\alpha,\beta} \ : \ \boxdot^{\beta} \square \ \square \quad \text{and} \quad \mathsf{rel}(p_{\square}^{\alpha,\beta}) \equiv \boxdot^{\alpha}$$

The elements $p_{\square}^{\alpha,\beta}$ of $\boxdot^{\beta} \square \square$

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$\boxdot^{\beta} \square \square$ may have several inhabitants
A translation must explain which one to target.
We need to annotate $\square$ everywhere!

4

Trocq

Annotating

$$M, N, A, B \in \mathcal{T}_{CC_\omega^+} ::= \Box_i^\alpha \mid x \mid M\ N \mid \lambda x : A.\ M \mid \Pi x : A.\ B$$

$$\frac{\Gamma \vdash_+ M : A \qquad \Gamma \vdash_+ A \preccurlyeq B}{\Gamma \vdash_+ M : B}\ (\textsc{Conv}^+)$$

$$\frac{(\alpha, \beta) \in \mathcal{D}_\Box}{\Gamma \vdash_+ \Box_i^\alpha : \Box_{i+1}^\beta}\ (\textsc{Sort}^+) \qquad \frac{(x, A) \in \Gamma \qquad \Gamma \vdash_+}{\Gamma \vdash_+ x : A}\ (\textsc{Var}^+) \qquad \frac{\Gamma \vdash_+ A : \Box_i \qquad x \notin \mathrm{Var}(\Gamma)}{\Gamma,\ x : A \vdash_+}\ (\textsc{Context}^+)$$

$$\frac{\Gamma \vdash_+ M : \Pi x : A.\ B \qquad \Gamma \vdash_+ N : A}{\Gamma \vdash_+ M\ N : B[x := N]}\ (\textsc{App}^+) \qquad \frac{\Gamma, x : A \vdash_+ M : B}{\Gamma \vdash_+ \lambda x : A.\ M : \Pi x : A.\ B}\ (\textsc{Lam}^+)$$

$$\frac{\Gamma \vdash_+ A : \Box_i^\alpha \qquad \Gamma \vdash_+ B : \Box_i^\beta \qquad \mathcal{D}_\rightarrow(\gamma) = (\alpha, \beta)}{\Gamma \vdash_+ A \rightarrow B : \Box_i^\gamma}\ (\textsc{Arrow}^+) \qquad \frac{\Gamma \vdash_+ A : \Box_i^\alpha \qquad \Gamma, x : A \vdash_+ B : \Box_i^\beta \qquad \mathcal{D}_\Pi(\gamma) = (\alpha, \beta)}{\Gamma \vdash_+ \Pi x : A.\ B : \Box_i^\gamma}\ (\textsc{Pi}^+)$$

Subtyping

$$\frac{\Gamma \vdash_{+} A : K \qquad \Gamma \vdash_{+} B : K \qquad A \equiv B}{\Gamma \vdash_{+} A \preccurlyeq B}\ (\textsc{SubConv})$$

$$\frac{\alpha \geq \beta \qquad i \leq j}{\Gamma \vdash_{+} \Box_i^{\alpha} \preccurlyeq \Box_j^{\beta}}\ (\textsc{SubSort})$$

$$\frac{\Gamma \vdash_{+} M'\ N : K \qquad \Gamma \vdash_{+} M \preccurlyeq M'}{\Gamma \vdash_{+} M\ N \preccurlyeq M'\ N}\ (\textsc{SubApp})$$

$$\frac{\Gamma, x : A \vdash_{+} M \preccurlyeq M'}{\Gamma \vdash_{+} \lambda x : A.\, M \preccurlyeq \lambda x : A.\, M'}\ (\textsc{SubLam})$$

$$\frac{\Gamma \vdash_{+} \Pi x : A.\, B : \Box_i \qquad \Gamma \vdash_{+} A' \preccurlyeq A \qquad \Gamma, x : A' \vdash_{+} B \preccurlyeq B'}{\Gamma \vdash_{+} \Pi x : A.\, B \preccurlyeq \Pi x : A'.\, B'}\ (\textsc{SubPi})$$

$$K ::= \Box_i \mid \Pi x : A.\, K$$

Calculus for Trocq

$$\frac{(\alpha,\beta) \in \mathcal{D}_{\square}}{\Delta \vdash_t \square_i^{\alpha} \ @\ \square_{i+1}^{\beta} \sim \square_i^{\alpha} \because p_{\square_i}^{\alpha,\beta}} \ (\textsc{TrocqSort}) \qquad \frac{(x, A, x', x_R) \in \Delta \quad \ldots}{\Delta \vdash_t x \ @\ A \sim x' \because x_R} \ (\textsc{TrocqVar})$$

$$\frac{\Delta \vdash_t M \ @\ \Pi x : A.\, B \sim M' \because M_R \qquad \Delta \vdash_t N \ @\ A \sim N' \because N_R}{\Delta \vdash_t M\ N \ @\ B[x := N] \sim M'\ N' \because M_R\ N\ N'\ N_R} \ (\textsc{TrocqApp})$$

$$\frac{\Delta \vdash_t A \ @\ \square_i^{\alpha} \sim A' \because A_R \qquad \Delta, x \ @\ A \sim x' \because x_R \vdash_t M \ @\ B \sim M' \because M_R}{\Delta \vdash_t \lambda x : A.\, M \ @\ \Pi x : A.\, B \sim \lambda x' : A'.\, M' \because \lambda x\, x'\, x_R.\, M_R} \ (\textsc{TrocqLam})$$

$$\frac{\Delta \vdash_t A \ @\ \square_i^{\alpha} \sim A' \because A_R \qquad \Delta \vdash_t B \ @\ \square_i^{\beta} \sim B' \because B_R \qquad (\alpha,\beta) = \mathcal{D}_{\rightarrow}(\delta)}{\Delta \vdash_t A \rightarrow B \ @\ \square_i^{\delta} \sim A' \rightarrow B' \because p_{\rightarrow}^{\delta}\ A_R\ B_R} \ (\textsc{TrocqArrow})$$

$$\frac{\Delta \vdash_t A \ @\ \square_i^{\alpha} \sim A' \because A_R \qquad \Delta, x \ @\ A \sim x' \because x_R \vdash_t B \ @\ \square_i^{\beta} \sim B' \because B_R \qquad (\alpha,\beta) = \mathcal{D}_{\Pi}(\delta)}{\Delta \vdash_t \Pi x : A.\, B \ @\ \square_i^{\delta} \sim \Pi x' : A'.\, B' \because p_{\Pi}^{\delta}\ A_R\ B_R} \ (\textsc{TrocqPi})$$

$$\frac{\Delta \vdash_t M \ @\ A \sim M' \because M_R \qquad \gamma(\Delta) \vdash_{+} A \preccurlyeq B}{\Delta \vdash_t M \ @\ B \sim M' \because \Downarrow_B^A M_R} \ (\textsc{TrocqConv})$$

Abstraction Theorem for Trocq

We have:

$$\frac{\gamma(\Delta) \vdash_+ \qquad \gamma(\Delta) \vdash_+ M : A \qquad \Delta \vdash_t M \mathbin{@} A \sim M' \because M_R \qquad \Delta \vdash_t A \mathbin{@} \square_i^\alpha \sim A' \because A_R}{\gamma(\Delta) \vdash_+ M' : A' \qquad \text{and} \qquad \gamma(\Delta) \vdash_+ M_R : \mathrm{rel}(A_R)\ M\ M'}$$

Remark A:

$$\frac{\gamma(\Delta) \vdash_+ A : \square^\alpha \qquad \Delta \vdash_t A \mathbin{@} \square^\alpha \sim A' \because A_R}{\gamma(\Delta) \vdash_+ A_R : \boxdot^\alpha\ A\ A'}.$$

Remark B:

$$\vdash_+ p_\square^{\alpha,\beta} : \boxdot^\beta\ \square^\alpha\ \square^\alpha$$

5

Extra material

The elements p_{Π}^{δ} of $\boxdot^{\delta}\ (\Pi A.B)\ (\Pi A'.B')$

We need to identify the triples $(\alpha, \beta, \delta) \in \mathcal{A}^3$ for which it is possible to construct a term p_{Π}^{δ} such that:

$$\frac{\delta \vdash A_R\ :\ \boxdot^{\alpha}\ A\ A' \qquad \delta,\ x : A,\ x' : A',\ x_R : A_R\ x\ x' \vdash B_R\ :\ \boxdot^{\beta}\ B\ B'}{\delta\ \vdash\ p_{\Pi}^{\delta}\ A_R\ B_R\ :\ \boxdot^{\delta}\ (\Pi x : A.\, B)\ (\Pi x' : A'.\, B')} \quad \text{and}$$

$$\mathsf{rel}(p_{\Pi}^{\delta}\ A_R\ B_R) \equiv \lambda f.\lambda f'.\Pi(x\ :\ A)(x'\ :\ A')(x_R : \mathsf{rel}(A_R)\ x\ x').$$
$$\mathsf{rel}(B_R)\ (f\ x)\ (f\ x')$$

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$$\frac{\delta \vdash A_R\ :\ \boxdot^{\alpha}\ A\ A' \qquad \delta,\ x : A,\ x' : A',\ x_R : A_R\ x\ x' \vdash B_R\ :\ \boxdot^{\beta}\ B\ B'}{\delta\ \vdash\ p_{\Pi}^{\delta}\ A_R\ B_R\ :\ \boxdot^{\delta}\ (\Pi x : A.\ B)\ (\Pi x' : A'.\ B')} \quad \text{and}$$

$$\mathrm{rel}(p_{\Pi}^{\delta}\ A_R\ B_R) \equiv \lambda f.\lambda f'.\Pi(x\ :\ A)(x'\ :\ A')(x_R : \mathrm{rel}(A_R)\ x\ x').$$
$$\mathrm{rel}(B_R)\ (f\ x)\ (f\ x')$$

We prove that p_{Π}^{δ} exists for all $(\alpha, \beta) \in \mathcal{D}_{\pi}(\delta)$, where ...

Definition of $\mathcal{D}_{\Pi}(\delta)$

For any $\delta \in \mathcal{A}$:

$$\mathcal{D}_{\Pi}(\delta) = \mathcal{D}_{\Pi}(\delta_1, 0) \vee \mathcal{D}_{\Pi}(\delta_2, 0)^{-1}$$

Where for all $\alpha, \beta \in \mathcal{A}$

$$\begin{aligned}
(\alpha, \beta)^{-1} &\triangleq (\alpha^{-1}, \beta^{-1}) \\
\alpha^{-1} &\triangleq (\alpha_2, \alpha_1) \\
(\alpha, \beta) \vee (\alpha', \beta') &\triangleq (\alpha \vee \alpha', \beta \vee \beta') \\
\alpha \vee \beta &\triangleq (\alpha_1 \vee \beta_1, \alpha_2 \vee \beta_2)
\end{aligned}$$

Definition of $\mathcal{D}_{\Pi}(\delta)$

For any $\delta \in \mathcal{A}$:

$$\mathcal{D}_{\Pi}(\delta) = \mathcal{D}_{\Pi}(\delta_1, 0) \vee \mathcal{D}_{\Pi}(\delta_2, 0)^{-1}$$

Where for all $\alpha, \beta \in \mathcal{A}$

$$(\alpha, \beta)^{-1} \triangleq (\alpha^{-1}, \beta^{-1})$$
$$\alpha^{-1} \triangleq (\alpha_2, \alpha_1)$$
$$(\alpha, \beta) \vee (\alpha', \beta') \triangleq (\alpha \vee \alpha', \beta \vee \beta')$$
$$\alpha \vee \beta \triangleq (\alpha_1 \vee \beta_1, \alpha_2 \vee \beta_2)$$

Thus, it suffices to define $\mathcal{D}_{\Pi}(m, 0)$ for all $m \in \{0, 1, 2_a, 2_b, 3, 4\}$
The same holds for $\mathcal{D}_{\rightarrow}(\delta)$.

Definition of $\mathcal{D}_{\Pi}(m,0)$ and $\mathcal{D}_{\rightarrow}(m,0)$

m	$\mathcal{D}_{\Pi}(m,0)_1$	$\mathcal{D}_{\Pi}(m,0)_2$
0	$(0,0)$	$(0,0)$
1	$(0,2_a)$	$(1,0)$
2_a	$(0,4)$	$(2_a,0)$
2_b	$(0,2_a)$	$(2_b,0)$
3	$(0,4)$	$(3,0)$
4	$(0,4)$	$(4,0)$

m	$\mathcal{D}_{\rightarrow}(m,0)_1$	$\mathcal{D}_{\rightarrow}(m,0)_2$
0	$(0,0)$	$(0,0)$
1	$(0,1)$	$(1,0)$
2_a	$(0,2_b)$	$(2_a,0)$
2_b	$(0,2_a)$	$(2_b,0)$
3	$(0,3)$	$(3,0)$
4	$(0,4)$	$(4,0)$

6

Conclusion

Comparison

	Magaud [10]	Setoid rw. [12]	CoqEAL [5]	Transfer [6–9]	ZH [15]	TTS [14]	ACMZ [1]	Trackt [3]	Trocq
Heterogeneous relations	✓	✗	✓	✓	✓	✓	✓	✓	✓
Internal	✗	✓	✓	✓	✓	✓	✓	✓	✓
No anticipation	✓	✓	✓	✓	✓	✓	✗	✓	✓
Substitution under $\forall$	✓	✓	✗	✓	✓	✓	✓	✓	✓
Substitution in dep. types	✓	✗	✗	✗	✗	✓	✓	✗	✓
No univalence for ?	✓	✓	✓	✓	✓	✗	✗	✓	✓
Preorder relations	✗	✓	?	?	?	✗	?	?	✏
Subrelations	✗	✓	✗	✗	✗	✗	✗	✗	✏
QERs	✗	✏	➡	➡	➡	✗	✓	✗	➡
Subtyping relations	✗	✗	➡	➡	➡	✗	✗	➡	➡
System	Coq	Coq	Coq	Isab	Coq	Coq	(Cub	Coq	Coq

Bring home

- Change representation **without univalence** in some cases.
- Change representation **with partial isos** in some cases.

In our current version,

- univalence is required if and only if there is some $\square^\alpha$ such that $\alpha \geq (2_b, 0)$ or $\alpha \geq (0, 2_b)$ occurs in the derivation.
- reducing a goal G to an hypothesis H corresponds to finding an element $\boxdot^{(0,1)}\ G\ H$ (i.e. an arrow $H \rightarrow G$). If the body of G and H have the right variance, we might keep the invariant that nothing more than the partial isos $\square^{(4,2_a)}$, $\square^{(4,2_b)}$,$\square^{(2_a,4)}$ or $\square^{(2_b,4)}$ are required on given types.

In the future (with a bit more work), we may unify

- CoqEAL
- Univalent paramericity
- Generalized (Setoid) rewriting

7

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